THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 3

1. Draw the following subsets of $\mathbb{R}^2$.

   (a) $D = \{(x, y) : 0 \leq x \leq y\}$;

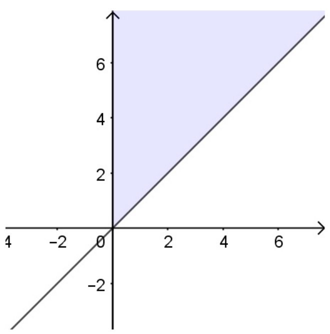
   (b) $D = \{(x, y) : x - y > 0\}$;

   (c) $D = \{(x, y) : xy \geq 0\}$;

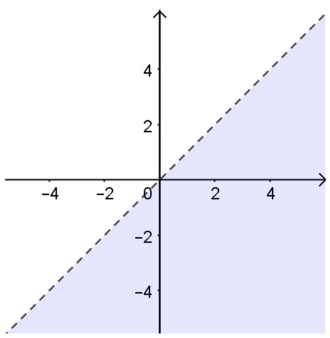
   (d) $D = \{(x, y) : |x| + |y| < 1\}$.
   (Hint: Write down the equation $|x| + |y| = 1$ explicitly in every quadrant.)

   Ans:

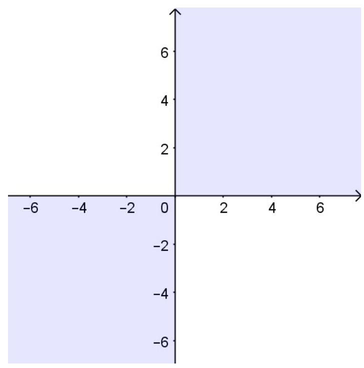
   (a)



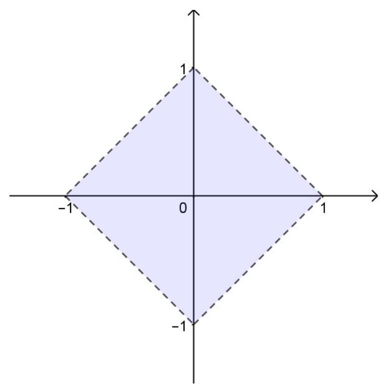
   (b)



   (c)



   (d)



2. Describe the following subsets of $\mathbb{R}^2$.

   (a) $D = \{(r, \theta) : 1 < r < 2\}$;

   (b) $D = \{(r, \theta) : 0 \leq r \leq 3, 0 \leq \theta \leq \pi\}$.

   Ans:

   (a) D is an open annulus where the inner and outer radius are 1 and 2 respectively.

   (b) D is the upper half disk with radius 3.

3. Match the following polar equations and curves.

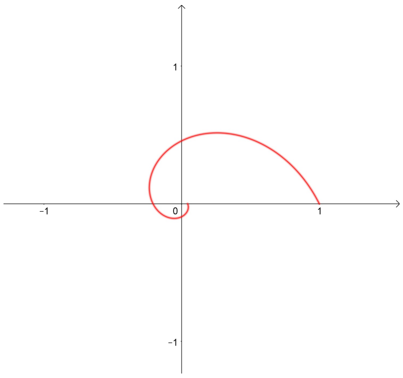
   (a) $r = \cos 2\theta$ for $0 \leq \theta \leq 2\pi$;

(b) $r = \sin 2\theta$ for $0 \leq \theta \leq 2\pi$;

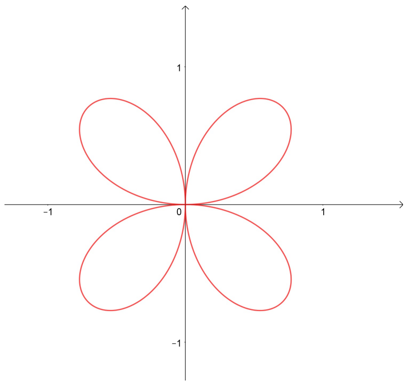
(c) $r = e^{-\theta/2}$ for $0 \leq \theta \leq 2\pi$;

(d) $r = \dfrac{1 - \cos\theta}{2}$ for $0 \leq \theta \leq 2\pi$.

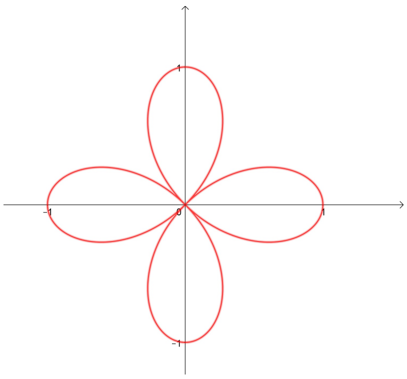
(i)



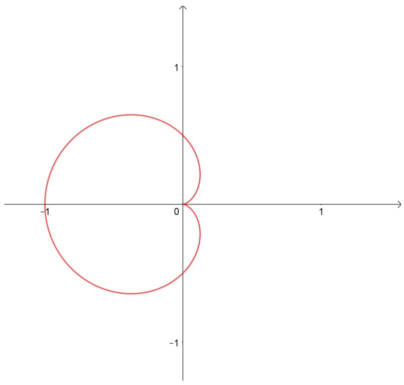
(iii)



(ii)



(iv)



Ans:

(a) (ii) (b) (iii) (c) (i) (d) (iv)

4. Let $S = \{(x, 0) \in \mathbb{R}^2 \,:\, x \in \mathbb{R}\}$. Show that

   (a) $\text{Int}(S) = \phi$;

   (b) $\partial S = S$;

   (c) $\text{Ext}(S) = \{(x, y) \in \mathbb{R}^2 \,:\, x \in \mathbb{R},\, y \neq 0\}$.

   Ans:

   (a) Suppose that $(a, b) \in \mathbb{R}^2$ is an interior point of S. Then there exists $r > 0$ such that $B_r(a, b) \subset S$.
   Consider two points $(a, b + r/2)$ and $(a, b - r/2)$ in $B_r(a, b)$, at least one of $b + r/2$ and $b - r/2$ is nonzero.
   That means at least one of the points is not in S, which is a contradiction.
   Therefore, $\text{Int}(S) = \phi$.

   (b)
   - Firstly, we claim $S \subset \partial S$.
     Let $(x, 0) \in S$. Then for all $r > 0$, we have $(x, 0) \in B_r(x, 0) \cap S$ and $(x, r/2) \in B_r(x, 0) \cap (\mathbb{R}^2 \backslash S)$.
     Therefore, both $B_r(x, 0) \cap S$ and $B_r(x, 0) \cap (\mathbb{R}^2 \backslash S)$ are nonempty which implies $(x, 0) \in \partial S$.
   - Secondly, we claim any point in $\mathbb{R}^2 \backslash S$ is not a boundary point of S.
     Let $(x, y) \in \mathbb{R}^2 \backslash S$, where $y \neq 0$. We let $r = |y|/2$, we can see that $B_r(x, y) \subset \mathbb{R}^2 \backslash S$.
     Therefore, $B_r(x, y) \cap S$ is empty, which implies (x, y) is not a boundary point of S.

   (c)
   - Firstly, we claim $\mathbb{R}^2 \backslash S \subset \text{Ext}\, S$.
     Let $(x, y) \in \mathbb{R}^2 \backslash S$, where $y \neq 0$. We let $r = |y|/2$, we can see that $B_r(x, y) \subset \mathbb{R}^2 \backslash S$.
     Therefore, $(x, y) \in \text{Ext}\, S$.
   - Secondly, let $(x, 0) \in S$. It is clear that for all $r > 0$, $(x, 0) \in B_r(x, 0)$ is a point in S.
     Therefore, $B_r(x, 0)$ is not a subset of $\mathbb{R}^2 \backslash S$, which implies $(x, 0)$ is not an exterior point of S.

5. Let $S = \{\frac{1}{n} \,:\, n \in \mathbb{Z}^+\}$ be a subset of $\mathbb{R}$.
   Write down $\text{Int}(S)$ and ∂S.

   Ans:

   $\text{Int}(S) = \phi$, $\partial S = \{\frac{1}{n} \,:\, n \in \mathbb{Z}^+\} \cup \{0\}$.

6. Let $S = \{(x, y) \in \mathbb{R}^2 : |x| \geq 1\}$ be a subset of $\mathbb{R}^2$.

   Show that S is not path connected.

   Ans:

   Suppose that S is a path connected set.

   Since $(-1, 0)$ and $(1, 0)$ are points in S, there exists a continuous curve $\gamma : [0, 1] \to S$ such that $\gamma(0) = (-1, 0)$ and $\gamma(1) = (1, 0)$.

   If we write $\gamma(t) = (x(t), y(t))$, then $x(t)$ is a continuous function with $x(0) = -1$ and $x(1) = 1$. By intermediate value theorem, there exists $t_0 \in (0, 1)$ such that $x(t_0) = 0$.

   Therefore, $\gamma(t_0) = (x(t_0), y(t_0)) = (0, y(t_0))$ which is a point lying on γ but not in S, which is a contradiction.

7. Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be a subset of $\mathbb{R}^2$.

   Show that S is a compact set.

   Ans:

   Firstly, we would like to show that S is closed, but it is equivalent to show that $\mathbb{R}^2 \backslash S$ is an open set.

   Let $(x_0, y_0) \in \mathbb{R}^2 \backslash S$. Then, we have $R = x_0^2 + y_0^2 > 1$.

   Let $r = \dfrac{R - 1}{2} > 0$. Then, $B_r(x_0, y_0) \subset \mathbb{R}^2 \backslash S$ which implies $\mathbb{R}^2 \backslash S$ is open.

   Clearly, S is bounded, so S is a compact subset in $\mathbb{R}^2$.

8. Let $S = \{(e^t \cos t, e^t \sin t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ be a subset of $\mathbb{R}^2$. Prove that

(a) S is unbounded;

(b) $\mathbf{0} = (0, 0)$ is a boundary point of S.

Ans:

(a) Let $M > 0$. By taking $t_0 \in \mathbb{R}$ such that $t_0 > \ln M$, we have $e^{t_0} > M$.

Then, consider $\mathbf{p} = (e^{t_0} \cos t_0, e^{t_0} \sin t_0) \in S$, we have $|\mathbf{p}| = \sqrt{(e^{t_0} \cos t_0)^2 + (e^{t_0} \sin t_0)^2} = e^{t_0} > M$.

Therefore, S is unbounded.

(b) Let $r > 0$. By taking $t_0 \in \mathbb{R}$ such that $t_0 < \ln r$, we have $e^{t_0} < r$.

Then, consider $\mathbf{p} = (e^{t_0} \cos t_0, e^{t_0} \sin t_0) \in S$, we have $|\mathbf{p}| = \sqrt{(e^{t_0} \cos t_0)^2 + (e^{t_0} \sin t_0)^2} = e^{t_0} < r$ and so $\mathbf{p} \in B_r(\mathbf{0}) \cap S$.

Also, consider $\mathbf{q} = (e^{t_0} \cos(t_0 + \frac{\pi}{2}), e^{t_0} \sin(t_0 + \frac{\pi}{2})) \in \mathbb{R}^2 \backslash S$, we have $|\mathbf{q}| = \sqrt{[e^{t_0} \cos(t_0 + \frac{\pi}{2})]^2 + [e^{t_0} \sin(t_0 + \frac{\pi}{2})]^2} = e^{t_0} < r$ and so $\mathbf{q} \in B_r(\mathbf{0}) \cap \mathbb{R}^2 \backslash S$.

Therefore, $\mathbf{0} = (0, 0)$ is a boundary point of S.